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Abstract

In this paper, maximal achievable rate regions are derived for power constrained AWGN broadcast channel involving finite constellations and two users. The achievable rate region is studied for various transmission strategies including superposition coding and compared to standard schemes such as time sharing. The maximal achievable rates are obtained by optimizing over both the joint distribution of probability and over the constellation symbol positions. A numerical solution is proposed for solving this non-convex optimization problem. Then, we consider several variations of the same problem by introducing various constraints on the optimization variables. The aim is to evaluate efficiency vs complexity tradeoffs of several transmission strategies, some of which (the simplest ones) can be found in actual standards. The improvement for each scheme is evaluated in terms of SNR savings for target achievable rates or/and percentage of gain in achievable rates for one user compared to a reference scheme. As an application, two scenarios of coverage areas and user alphabets are considered. This study allows to evaluate with practical criteria the performance improvement brought by more advanced schemes.

Keywords

AWGN broadcast channels, achievable rate region, hierarchical modulation, superposition modulation, superposition coding, constellation shaping, non-

convex optimization.

1 Introduction

During the past few decades, information networks have witnessed tremendous and rapid advances, based on the important growth in the adoption of new wireless technologies, applications and services, first from cellular networks and more recently for computer networks (WLANs). Consequently, wireless networks are exposed to capacity and coverage problems and the focus is now shifting towards capturing some of the aspects of realistic networks by studying natural network models such as models with broadcasting.

In 1972, achievable rate region is obtained by Cover in [1] for Gaussian broadcast channels with two outputs and generalized by Bergmans to broadcast channels with any number of outputs [2]. Roughly a year later, the optimality of the sets of achievable rates was established by Bergmans [3] and Gallager [4]. Superposition coding is a possible solution to achieve good rate regions in which information intended for high-noise receivers and information intended for low-noise receivers are superimposed and transmitted simultaneously on the same radio resource. The low-noise receivers can always decode messages intended for the high-noise receivers. Thus they effectively cancel out the interference due to the signal intended for the high-noise receivers, and then decode their own message. The high-noise receivers decodes its message by treating the low-noise receivers message as noise. Superposition coding appears in several contexts in information theory and is closely related to multilevel coding and unequal error protection [5], [6]. Cover showed [1] that superposition coding reaches the theoretical limit of the capacity region for two user Gaussian broadcast channel using an infinite Gaussian input alphabet for each user. A treatment of the case of multiple transmitter/receivers for the band-limited additive white Gaussian noise channel is given by Bergmans and Cover in [7] where it is proved that superposition coding can achieve higher rate region than orthogonal schemes such as frequency-division multiple access (FDMA) or time-division multiple access (TDMA). However, in actual transmissions systems, the channel input is constrained to a finite size alphabet with equal probability symbols. A well known practical implementation of superposition coding is hierarchical modulation, also called layered modulation, which uses constellations with non-uniformly spaced signal points creating different levels of error protection. Hierarchical modulation is used to mitigate the cliff effect in digital television broadcast and is included in various standards, such as Digital Video Broadcast for Terrestrial Television (DVB-T) [8], DVB to Handhelds (DVB-H) and DVB Satellite services to Handhelds (DVB-SH) [9] standard proposal for mobile digital TV transmission. A study about the performance of hierarchical modulation and a comparison with time sharing strategy in terms of achievable rates can be found in [10].

The restriction imposed by practical systems in using finite signaling constellation and equiprobable symbols reduces the achievable rates and leads to a gap with the capacity region achieved with Gaussian input alphabets for AWGN broadcast channel. This gap can be reduced using a technique called constellation shaping. In fact, most results for constellation shaping with finite signal constellations consider only point to point communication systems [11]. Then the concept of constellation shaping has been adapted to most modern coding and modulation techniques as for example turbo-coding and BICM schemes [12]-[19]. For broadcast channels, the achievable rate region for two-user AWGN broadcast channels with finite input alphabets is derived in [20] when superposition of modulated signal is used as transmission strategy. In their work, the authors assume a uniform distribution over the finite input set. To our knowledge, no study is available about the maximization of the achievable rate region for two-user AWGN broadcast channels with finite size constellations by optimizing over both the joint probability distribution and constellation symbol positions for a broadcast transmission strategy. This general framework encompasses hierarchical modulations as a special case. In this paper, maximal achievable rate regions are derived for power constrained AWGN broadcast channel of two users with M-Pulse Amplitude Modulation (M-PAM) constellations of M points using various transmission strategies. A numerical solution is proposed for solving this non-concave optimization problem. In a typical broadcast system, there is a trade off between achievable rates and coverage areas. Therefore, we are interested in determining the transmission strategy which provides the best achievable rates or the maximal SNR gain for a given coverage scenario. The compromise between simplicity of implementation and expected gains is also evaluated.

The organization of the paper is as follows. Section 2 recalls some information theory results on broadcast channels and degraded broadcast channels. In section 3 various transmission strategies for broadcast systems are described. Section 4 gives a formulation of the problem in terms of optimization for the various transmission strategies under consideration. Then computational aspects are discussed. An iterative algorithm is proposed for the computation of maximal achievable rate regions using superposition coding (general case) and M-PAM constellation or in the particular case of superposition modulation. The proposed algorithm can handle an optimization with respect to the joint distribution of probability or with respect to the positions of constellation symbols. Both variables can also be considered jointly. Obviously, the best results are obtained for the most general case. Our target is to: *(i)* evaluate the loss experienced by using simple schemes, *(ii)* identify situations in which complex schemes (non-standard) lead to significant improvements. As an application, we consider, in section 5, several scenarios of coverage areas and user alphabets and we give conclusions about the transmission strategies which can provide the best trade off between efficiency and complexity of implementation.

2 AWGN Broadcast Channels

A two-receiver (users) broadcast channel (BC) consists of an input alphabet $\mathcal{X}$, two outputs alphabets $\mathcal{Y}_1$ (user 1), $\mathcal{Y}_2$ (user 2) and a conditional pdf $P_{Y_1Y_2|X}$ on $\mathcal{Y}_1 \times \mathcal{Y}_2$. Let X, Y_1 and Y_2 be random variables representing the input and outputs of the BC. Figure 1 depicts the two users BC with two independent messages W_1 and W_2. The encoder generates a codeword $x^n(w_1, w_2)$ of length n based on these two messages. Each user receives respectively y_1^n and y_2^n. A BC is said to be physically degraded if $P_{Y_1Y_2|X}(y_1, y_2|x) = P_{Y_1|X}(y_1|x) \cdot P_{Y_2|Y_1}(y_2|y_1)$ (i.e. $X \to Y_1 \to Y_2$ form a Markov chain). A BC is said to be stochastically degraded or degraded if there exists a random variable $\tilde{Y}_1$ which has the same conditional pdf as Y_1 given X such that $X \to \tilde{Y}_1 \to Y_2$ form a Markov chain. We are interested in degraded BC because its capacity region is known, while it is not available for the general case.

In our system model, W_1 denotes the private message intended for receiver 1 only and W_2 is a common message for both receivers. A typical example of this situation is digital TV broadcasting to two different groups of receivers, classified according to their channel conditions, where the basic signal (common signal) should be available to all receivers. The higher quality is realized by adding the basic signal with an incremental signal (private signal for receivers of good channel conditions) which carries TV signal with a high data rate, such as HDTV.

Let R_1 and R_2 be the rates at which the transmitter is sending W_1 and W_2 respectively. Thus user 1 achieves $R_1 + R_2$ while user 2 achieves R_2. The capacity region of the degraded broadcast channel $X \to Y_1 \to Y_2$ in figure 1 is the convex hull of the closure of rate pairs $(R_1 + R_2, R_2)$ satisfying:

$$R_1 \leq I(X; Y_1|U) \tag{1}$$

$$R_2 \leq I(U; Y_2) \tag{2}$$

for some joint distribution $P_{UXY_1Y_2} = P_{UX} \cdot P_{Y_1|X} \cdot P_{Y_2|X}$ on $\{\mathcal{U} \times \mathcal{X} \times \mathcal{Y}_1 \times \mathcal{Y}_2\}$ [21]. $P_{Y_1|X}$ and $P_{Y_2|X}$ are conditional pdfs that depend on the channel model. P_{UX} is the joint probability distribution of U and X, where the auxiliary random variable U has cardinality bounded by $|\mathcal{U}| \leq \min\{|\mathcal{X}|, |\mathcal{Y}_1|, |\mathcal{Y}_2|\}$. The capacity region is achieved using superposition coding where U serves as the center of a cloud of codewords that can be distinguished by both receivers. Since the capacity region of a BC depends only on the conditional marginals, the capacity region of the stochastically degraded BC is equal to that of the corresponding physically degraded channel. Cover [1] showed that in the case of binary symmetric BC and AWGN BC, superposition coding expands the rate region beyond that achievable with time sharing.

Now consider the Gaussian broadcast channel with two users. Without loss of generality, assume that Y_1 is less noisy than Y_2. It can easily be shown that scalar Gaussian broadcast channels are equivalent to a degraded channel,

$$Y_1 = X + Z_1 \tag{3}$$

$$Y_2 = X + Z_2 = Y_1 + Z_2' \tag{4}$$

where $Z_1 \sim \mathcal{N}(0, \sigma_1^2)$, $Z_2 \sim \mathcal{N}(0, \sigma_2^2)$, $Z_2' \sim \mathcal{N}(0, \sigma_2^2 - \sigma_1^2)$ and Z_1, Z_2' are independent. Thus Gaussian BC is stochastically degraded. We assume an average power constraint on the transmitted power P defined as $\mathbb{E}[X^2] \leq P$. The received signal to noise ratio for each user is $SNR_i = \frac{P}{\sigma_i^2}$, where $SNR_1 > SNR_2$ and σ_i^2 is the variance of the noise Z_i. The capacity region of the AWGN-BC is the set of rate pairs $(R_1 + R_2, R_2)$ such that:

$$R_1 \leq C(\alpha \cdot SNR_1) \tag{5}$$

$$R_2 \leq C(\frac{(1-\alpha) \cdot SNR_2}{\alpha \cdot SNR_2 + 1}) \tag{6}$$

for all $\alpha \in [0, 1]$, where $C(x) = \frac{1}{2} \cdot \log_2(1 + x)$. The theoretical limit of two-user AWGN BC is achieved by using signal superposition [1].

3 Broadcast transmission strategies

In this section, various transmission strategies for broadcast systems are described. The strategies are presented in ascending order of implementation complexity. Specifically, by moving from one strategy to another, we release some constraints on the system implementation to reach finally the most complex strategy that can be used to broadcast information for users. Obviously, since the simple schemes can be understood as adding constraints to the most general case, their are less efficient in terms of attainable rates.

3.1 Time Sharing (TS)

Time sharing has been widely used in broadcast systems as broadcast transmission strategy. In time sharing scheme, a percentage of time is used to send one message and the rest of the time is used to send another message. Thus it is practical to implement because the rate pairs can be achieved by strategies used for point to point channel and sharing the time between messages. As in previous works on broadcasting, this situation serves as a reference for the more advanced schemes. In this work, a time sharing scheme with standard constellation M-PAM (Fig.2) is considered when symbols are used with equal probability. A standard M-PAM constellation is defined as a constellation with M real symbols belonging to $\mathcal{X} = \{M - 1 - 2 \cdot (i - 1), \text{for } i = 1, ..., M\}$. During the time slot dedicated to send a message, only one data stream is sent using the entire set of constellation points. In classical implementations of time sharing, the conventional M-PAM symbols are equally spaced and used with equal probability.

3.2 Hierarchical Modulation (HM)

In two layers hierarchical modulation, constellation symbols are used to transmit two data streams simultaneously for two users [22][23]. Constellation symbols are usually chosen with the same probability but may be non-equally spaced. These symbols can be considered as the sum of two lower order modulations, one for each user. The modulation with higher power is used for the "bad" channel, the one with smallest power for the "good" channel. Hence, the encoding using hierarchical modulation can be separable for the two streams which is more practical.

This is explained here using 4-PAM as an example. Fig.3 shows the constellation diagram of a hierarchical 4-PAM with parameter $\ell = \ell_1/\ell_2$ used to determine the spacing between the groups of constellation points (clouds). ℓ is the ratio of the spacing between the groups to the spacing between individual points within a group. Standard values of ℓ are 1, 2 and 4. When ℓ increases, with a fixed total transmission power P, the two points from both sides of origin form a cloud. The location of a point within its cloud is regarded as the information for the "good" user. The other information, i.e. the number of the cloud in which the point is located is the information for the "bad" user. In this way, two separate data streams can be made available for transmission. Formally, we are still dealing with 4-PAM but, in the hierarchical interpretation, it is viewed as the combination of 2 BPSK modulations which have different robustness to noise. In other words, the service coverage areas differ in size for both users. The better-protected data stream is referred to as the High- Priority (HP) stream which is mapped in Fig.3 to the most significant bit. The other one, is referred to as the Low-Priority (LP) stream (Fig.3) and mapped in Fig.3 to the least significant bit. Receivers with good reception conditions can receive both streams, while those with poorer reception conditions may only receive the high priority stream considering the LP stream as noise. This corresponds to a specific labeling of the modulation.

3.3 Superposition Modulation (SM)

In superposition modulation [24], the M constellation points are used such that the labeling is separable, *i.e.* $M = M_1 M_2$, and that the M points are obtained by adding (in $\mathbb{R}$) two rv's X_1 and X_2, of cardinality M_1 and M_2 respectively ($M_1, M_2 \in \mathbb{N} \setminus \{0, 1\}$). Thus this scheme is with an enlarged set of feasible labelings than in the previous case [25],[26]. This leads also to $U \equiv X_2$ for superposition modulation because user 2 can distinguish only U.

This work studies several cases of superposition modulation. First, when the constellation symbols for each user are used with equal probability. This case will be denoted as $SM_{\mathcal{X},\overline{P_{UX}},\overline{P_X}}$. This is a practical case since the encoding of the messages is separable and symbols are used with equal probability as in real transmission systems. Then, the constraint of using equiprobable symbols is released, the symbols of user constellations can be dependent and used with non-equal probability (P_{UX} non-uniform). Thus the encoding here is done jointly for the two messages. This strategy will be denoted $SM_{\overline{\mathcal{X}},P_{UX},P_X}$ when the symbols take the values of a standard M-PAM and $SM_{\mathcal{X},P_{UX},P_X}$ otherwise. In the latter case, the symbol positions can take arbitrary values and will be considered as variables to be optimized. The definition of superposition modulation can be generalized using more general form for P_{UX} than the uniform case. In superposition modulation, 2^{nR_2} independent codewords $u^n = x^{(2)n}(w_2)$ of length n are generated according to P_U and for each of these codewords, 2^{nR_1} satellite codewords $v^n = x^{(1)n}(w_1)$ are generated and added to form codewords $x^n(w_1, w_2) = u^n + v^n$ according to $P_{X|U}$. Thus, the fine information v^n is superimposed on the coarse information u^n.

Note that the capacity region of Gaussian broadcast channel is achieved using this coding scheme and successive cancellation decoding where U ($\equiv X_2$) and V ($\equiv X_1$) are independent random variables following normal distributions. However, we do not assume here that U and V are independent. Consequently, for superposition modulation, P_{UX} takes a specific expression. As an example, consider an 8-PAM modulation. In that case, the transmitted signal at time k is the sum of the two users signals and is given by $x_k = x_k^{(1)} + x_k^{(2)}$ where $x_k^{(1)} \in \mathcal{X}_1$ and $x_k^{(2)} \in \mathcal{X}_2$ with $M_1 \cdot M_2 = 8$. Two configurations are possible either $M_2 = 4$ ($\mathcal{X}_1$ is a BPSK and $\mathcal{X}_2$ is a 4-PAM) or $M_2 = 2$ ($\mathcal{X}_1$ is a 4-PAM and $\mathcal{X}_2$ is a BPSK). In both cases, P_{UX} is a sparse matrix of size $M_2 \times M$ with expression

$$P_{UX} = \begin{bmatrix} p_{00} & p_{01} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{12} & p_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{24} & p_{25} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_{36} & p_{37} \end{bmatrix} \quad if\ M_1 = 2, M_2 = 4 \tag{7}$$

$$P_{UX} = \begin{bmatrix} p_{00} & p_{01} & p_{02} & p_{03} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{14} & p_{15} & p_{16} & p_{17} \end{bmatrix} \quad if\ M_1 = 4, M_2 = 2 \tag{8}$$

where $P_{UX}[i,j] = p_{i-1,j-1} = \Pr\{U = u_{i-1}, X = x_{j-1}\}$. In both cases, the number of elements to be computed is 8.

Note also that P_{UX} and $\mathcal{X}$ (of cardinality M) determine the labeling of the input signal constellation for a fixed labeling for $\mathcal{X}_1$ and $\mathcal{X}_2$ [25],[26]. Thus the information can be distinguished using the labeling. Consider for example a label l_k^u of $log_2(|\mathcal{X}_2|)$ binary labels for u_k and l_j^v of $log_2(|\mathcal{X}_1|)$ binary labels for v_j with $k \in \{0, .., |\mathcal{X}_2| - 1\}$ and $j \in \{0, .., |\mathcal{X}_1| - 1\}$. Obviously, the M symbols $x_i,\ i \in \{0, .., |\mathcal{X}| - 1\}$ carry $log_2(M)$ binary labels which are the concatenations of the labels of u_k and v_j such as $x_i = u_k + v_j$.

Part of this work on superposition modulation was presented in [25],[26],[27], where the achievable rate regions for $SM_{\mathcal{X},\overline{P_{UX}},\overline{P_X}}$ and $SM_{\mathcal{X},P_{UX},P_X}$ strategies are analyzed using a 4-PAM constellation in [25],[26] and for {4,8,16}-PAM constellations in [27]. In this work, the achievable rates are also derived for $SM_{\overline{\mathcal{X}},P_{UX},P_X}$ using {4,8,16}-PAM constellations.

3.4 Superposition Coding (SC)

Superposition coding is one of the basics of coding schemes in network information theory. This idea was first introduced by Cover in an information theoretic study of broadcast channels [1].

In superposition coding, the joint distribution of probability P_{UX} can take a more general form than in the case of superposition modulation. In this case the labeling cannot allow to distinguish between the common information and the private information for user 1, a fact which increases the decoder complexity. Indeed, since the auxiliary random variable U has cardinality bounded by $|\mathcal{U}| \leq \min\{|\mathcal{X}|, |\mathcal{Y}_1|, |\mathcal{Y}_2|\}$, we use the name general superposition coding or superposition coding simply to describe the case where $|\mathcal{U}| = \min\{|\mathcal{X}|, |\mathcal{Y}_1|, |\mathcal{Y}_2|\}$. For superposition coding and with M-PAM modulation, P_{UX} is an $M \times M$ matrix with elements $p_{i,j}$.

The basics of superposition coding are briefly recalled below; a detailed description is given in [28]. In this scheme, 2^{nR_2} sequences $u^n(w_2)$,$w_2 \in [1, 2^{nR_2}]$ each i.i.d.,are generated randomly and independently to represent the coarse message each according to $\prod_{i=1}^{n} p_U(u_i)$. For each auxiliary sequence $u^n(w_2)$, randomly and conditionally independently generate 2^{nR_1} sequences $x^n(w_1, w_2)$,$w_1 \in [1, 2^{nR_1}]$, each according to $\prod_{i=1}^{n} p_{X|U}(x_i|u_i(w_2))$ to represent the fine message w_1. Thus in superposition coding, the auxiliary random variable U serves as a cloud center for the information, distinguishable by both receivers. In this case, the decoding of information by users is based on large block joint typicality. This comes in contrast with the simpler cases where the message for user 2 was carried by the center of *modulation* clouds which imply a possible scalar detection.

The achievable rates for superposition coding will be studied for various strategies corresponding to different constraints on P_{UX} and/or $\mathcal{X}$.

An exhaustive list of all the strategies under consideration is given in table 1 where redundant configurations are omitted.

4 Achievable Rate Regions

For a two user Gaussian BC, the theoretical limit of the capacity region is achieved using Gaussian input alphabet for each user. However, practical implementation constraints impose the use of finite input alphabets, and the symbols are usually chosen with equal probability. These restrictions contribute to increase the gap between the capacity region achieved with infinite Gaussian inputs and the throughput obtained in practical situations. In this section, we are interested in computing the achievable rate region of power constrained AWGN BC when the transmitted signal is modulated using an M-PAM constellation, under the various situations described above. Since the last case (superposition coding) encompasses all previous ones as special cases, the corresponding optimization problems can be solved with the same strategy, which is detailed in this section.

4.1 Problem Formulation

Consider a two users memoryless AWGN broadcast channel ($SNR_1 > SNR_2$) with signal power constraint P. The channel input belongs to a finite set

Transmission	Variables	Constraints	Designation
SM	$\mathcal{X}$	Uniform distribution for P_{UX}	$SM_{\mathcal{X},\overline{P_{UX}},\overline{P_X}}$
SM	P_{UX} s.t. $\sum_{i,j} p_{i,j} = 1$	Symbol locations: M-PAM	$SM_{\overline{\mathcal{X}},P_{UX},P_X}$
SM	$\mathcal{X}$ P_{UX} s.t. $\sum_{i,j} p_{i,j} = 1$		$SM_{\mathcal{X},P_{UX},P_X}$
SC	P_{UX} s.t. $\sum_{i} p_{i,j} = \frac{1}{M}$	Symbol locations: M-PAM Uniform distribution for P_X	$SC_{\overline{\mathcal{X}},P_{UX},\overline{P_X}}$
SC	$\mathcal{X}$ P_{UX} s.t. $\sum_{i} p_{i,j} = \frac{1}{M}$	Uniform distribution for P_X	$SC_{\mathcal{X},P_{UX},\overline{P_X}}$
SC	P_{UX} s.t. $\sum_{i,j} p_{i,j} = 1$	Symbol locations: M-PAM	$SC_{\overline{\mathcal{X}},P_{UX},P_X}$
SC	$\mathcal{X}$ P_{UX} s.t. $\sum_{i,j} p_{i,j} = 1$		$SC_{\mathcal{X},P_{UX},P_X}$

Table 1: Strategies under consideration

$\mathcal{X} = \{x_0, ..., x_{M-1}\} \subset \mathbb{R}$ represented by an M-PAM constellation. Assume a symmetric input signal constellation with respect to the origin. Since $\mathcal{U}$ has cardinality bounded by $|\mathcal{U}| \leq \min\{|\mathcal{X}|, |\mathcal{Y}_1|, |\mathcal{Y}_2|\}$ and the output alphabet cardinality for an AWGN channel is infinite, we have $|\mathcal{U}| \leq |\mathcal{X}|$. Thus $|\mathcal{U}| \leq M$.

To determine the maximal achievable rate region using superposition coding, consider the case $|\mathcal{U}| = M$. For superposition modulation, we take into account the specificity on P_{UX} given in section 3.3. We also consider within the same framework the problem of maximizing the achievable rates under additional constraints on optimization variables (P_{UX} and $\mathcal{X}$): standard M-PAM symbols values, uniform distribution for P_{UX}, uniform distribution for P_X. The problem of maximizing the achievable rates under a specific situation is solved subject to a combination of constraints according to table 1. We recall that in this work, message w_2 is a common message to both receivers and w_1 is a private message to user 1. Thus the achievable rate region (R_2 vs. $R_1 + R_2$) can be obtained by solving the weighted sum rate ($\theta \cdot R_1 + (1-\theta) \cdot R_2$) maximization for $\theta \in [0, 0.5]$. Indeed, for $\theta = 0$, we maximize the common information rate R_2 and when $\theta = 0.5$ we maximize the rate achieved by user 1 ($R_1 + R_2$). Using (1) and (2), the optimization problem under consideration is:

$$\begin{array}{lll} \max_{P_{UX},\mathcal{X}} & \theta \cdot I(X; Y_1|U) + (1-\theta) \cdot I(U; Y_2) \\ s.t. & p_{ij} \geq 0 \quad \forall i, j \\ & \sum_{i,j} p_{ij} \cdot x_j^2 \leq P \end{array} \tag{9}$$

and subject to the constraint on the joint pdf P_{UX} or on $\mathcal{X}$ given in table 1 for each strategy, where $p_{ij} = \Pr\{U = u_i, X = x_i\}$, $j \in \{0, .., M-1\}$ and $i \in \{0, .., |\mathcal{U}| - 1\}$. The two mutual information $I(X; Y_1|U)$ and $I(U; Y_2)$ can be written as follows

$$I(X;Y_1|U) = \sum_{i,j} \int_{-\infty}^{+\infty} p_{ij} P_{Y_1|X}(y_1|x_j) \log \frac{(\sum_{j'} p_{ij'}) P_{Y_1|X}(y_1|x_j)}{\sum_{j'} p_{ij'} P_{Y_1|X}(y_1|x_{j'})} dy_1 \quad (10)$$

$$I(U;Y_2) = \sum_{i} \int_{-\infty}^{+\infty} (\sum_{j} p_{ij} P_{Y_2|X}(y_2|x_j)) \log \frac{\sum_{j'} p_{ij'} P_{Y_2|X}(y_2|x_{j'})}{(\sum_{j'} p_{ij'})(\sum_{i',j'} p_{i'j'} P_{Y_2|X}(y_2|x_{j'}))} dy_2 \quad (11)$$

where all logarithms are taken base 2. The AWGN channel for each user is characterized by the conditional pdf

$$P_{Y_i|X}(y|x) = \frac{1}{\sqrt{2\pi\sigma_i^2}} . e^{-\frac{(y-x)^2}{2\sigma_i^2}} \quad i \in \{1,2\} \quad (12)$$

When $\theta = 0$ or $\theta = 1$ and for $|\mathcal{U}| = M$ (which are referred in this paper as point-to-point (PtP) channel case), the individual achievable rates R_2 and R_1 are maximized respectively. The problem (9) is equivalent to

$$\begin{array}{ll} \max_{P_X,\mathcal{X}} & I(X;Y_k) \\ s.t. & p_i \geq 0 \quad \forall i \\ & \sum_i p_i = 1 \\ & \sum_i p_i \cdot x_i^2 \leq P \end{array} \quad (13)$$

where $p_i = \Pr\{X = x_i\}$, $i \in \{0,..,M-1\}$ is the input probability distribution and $k \in \{1,2\}$. When $\theta = 0$ or 1, problem (13) is solved for $k = 2$ and 1 respectively with $I(X;Y_k)$ given by

$$I(X;Y_k) = \int_{-\infty}^{+\infty} \sum_{j} p_j P_{Y_k|X}(y_k|x_j) \log \frac{P_{Y_k|X}(y_k|x_j)}{\sum_{j'} p_{j'} P_{Y_k|X}(y_k|x_{j'})} dy_k \quad (14)$$

For the time sharing scheme using standard constellation, the achievable rate pair $(R_1 + R_2, R_2)$ is such that [1]:

$$\begin{cases} R_1 = \alpha \overline{R_1} \\ R_2 = (1-\alpha)\overline{R_2} \end{cases} \quad (15)$$

where $\overline{R_1}$ and $\overline{R_2}$ are achievable rates for PtP channel using standard M-PAM constellation at SNR_1 and SNR_2 respectively. Varying α from 0 to 1 yields achievable rate region.

Problem (9) is not convex, therefore direct numerical optimization is inefficient. Clearly, an exhaustive search is not feasible as the complexity would be exponential in the total number of variables. An iterative method for solving (9) is proposed in the next section.

4.2 Numerical solution

Consider a regularized version of (9) as:

$$L(P_{UX}, x_0, .., x_{M-1}, s) = \theta \cdot I(X; Y_1|U) + (1-\theta) \cdot I(U; Y_2) + s \cdot (P - \sum_{i=0}^{|\mathcal{U}|-1} \sum_{j=0}^{M-1} p_{ij} \cdot x_j^2) \tag{16}$$

where s is a regularization parameter. For a given value of s, the optimization problem in (16) is solved (for the most general case) with respect to P_{UX} and to $\mathcal{X} = (x_0, x_1, ..., x_{M-1})$ alternately until convergence:

$$P_{UX}^{(\ell)} = \arg \max_{P_{UX} \in \mathcal{C}} L(P_{UX}, x_0^{(\ell-1)}, .., x_{M-1}^{(\ell-1)}, s) \tag{17}$$

$$\mathcal{X}^{(\ell)} = \arg \max_{\mathcal{X}} L(P_{UX}^{(\ell)}, x_0, .., x_{M-1}, s) \tag{18}$$

where ℓ is the iteration index and $\mathcal{C}$ denotes the set of constraints on P_{UX} and can be defined either as $\mathcal{C} = \{P_{UX} : p_{ij} \geq 0, \sum_{i,j} p_{i,j} = 1\}$ or as $\mathcal{C} = \{P_{UX} : p_{ij} \geq 0, \sum_i p_{i,j} = \frac{1}{M}\}$ (equiprobable symbols). The optimization problem in (17) with constraint set $\mathcal{C} = \{P_{UX} : p_{ij} \geq 0, \sum_{i,j} p_{i,j} = 1\}$ can be handled by a modified Blahut-Arimoto type algorithm [29]. Indeed, in order to take into account the regularization, we can show that the "Blahut Arimoto "-type algorithm proposed in [30] for broadcast channels should be modified by replacing equation (19) of lemma 3 in [30] by $q^*(u,x) = \frac{\beta[Q,\tilde{Q},\bar{Q}](u,x) \cdot e^{-s\frac{x^2}{1-\theta}}}{\sum_{u',x'} \beta[Q,\tilde{Q},\bar{Q}](u',x') \cdot e^{-s\frac{x'^2}{1-\theta}}}$ instead of $q^*(u,x) = \frac{\beta[Q,\tilde{Q},\bar{Q}](u,x)}{\sum_{u',x'} \beta[Q,\tilde{Q},\bar{Q}](u',x')}$ where $\beta[Q,\tilde{Q},\bar{Q}](u,x)$ is defined in equation (19) of [30]. When there is an additional constraint on constellation symbols to be equiprobable *i.e.* $\mathcal{C} = \{P_{UX} : p_{ij} \geq 0, \sum_{i,j} p_{i,j} = 1 \;\text{ and }\; \sum_i p_{i,j} = \frac{1}{M}\}$, the "Blahut Arimoto "-type algorithm in [30] should also be modified to take into account the additional constraint. In this case, equation (19) of lemma 3 in reference [30] should be replaced by $q^*(u,x) = \frac{1}{|\mathcal{X}|} \cdot \frac{\beta[Q,\tilde{Q},\bar{Q}](u,x)}{\sum_u \beta[Q,\tilde{Q},\bar{Q}](u,x)}$, which does not depend on s, where $\beta[Q,\tilde{Q},\bar{Q}](u,x)$ is defined in equation (19) in this reference.

Now consider (18). The function $L(P_{UX}^{(\ell)}, x_0, .., x_{M-1}, s)$ is not a concave function for all $\mathcal{X} \in \mathbb{R}^M$. However, we observed in our experiments that $L(P_{UX}^{(\ell)}, x_0, .., x_{M-1}, s)$ is a concave function if $\mathcal{X} \in \mathcal{D}$ where $\mathcal{D} = \{\mathcal{X} \in \mathbb{R}^M : |x_i - x_j| > d \;\; \forall i,j \in \{0, .., M-1\}$ and $i \neq j\}$ and d depends on the size of the constellation and on the *SNR*. Since we are interested in finding non degenerated constellation, we restrict the optimization process to $\mathcal{D}$. Then a simplex method is used to perform the optimization with initial value in $\mathcal{D}$.

The alternative maximization method can at least increase the objective function in each iteration. In the experiments, we have observed that this method converges at least to a local maximum (denoted $p_{i,j}^*(s)$, $x_j^*(s)$, $0 \leq j \leq M-1$, $0 \leq i \leq |\mathcal{U}| - 1$).

Step 0	$s \leftarrow s^{(0)}$		
Step k	Step 0	$\mathcal{X} \leftarrow \mathcal{X}^{(0)}$ where $\mathcal{X} = (x_0, x_1, .., x_{M-1})$	
	Step ℓ	$P_{UX}^{(\ell)} = \arg\max_{P_{UX} \in \mathcal{C}} L(P_{UX}, \mathcal{X}^{(\ell-1)}, s^{(k-1)})$ $(P1)$ $\mathcal{X}^{(\ell)} = \arg\max_{\mathcal{X}} L(P_{UX}^{(\ell)}, \mathcal{X}, s^{(k-1)})$ $(P2)$	
	Stopping criterion	$\lvert L(P_{UX}^{(\ell)}, \mathcal{X}^{(\ell)}, s^{(k)}) - L(P_{UX}^{(\ell-1)}, \mathcal{X}^{(\ell-1)}, s^{(k-1)}) \rvert \leq \epsilon_L$	
	$s^{(k)} = [s^{(k-1)} - \beta(P - \sum_{i,j} p_{ij}^*(s^{(k-1)}) \cdot (x_j^*(s^{(k-1)}))^2)]^+$ where $[.]^+ = \max(., 0)$		
Stopping criterion	$\lvert s^{(k)} - s^{(k-1)} \rvert \leq \epsilon_s$		

Table 2: Numerical solution for solving (9)

We discuss now the choice of s. Since we do not know a priori which value of s may correspond to the satisfaction of the equality power-constraint, we propose to use an iterative process as follows:

$$s^{(k+1)} = \left[s^{(k)} - \gamma \cdot \left(P - \sum_{i=0}^{|\mathcal{U}|-1} \sum_{j=0}^{M-1} p_{ij}^*(s^{(k)}) \cdot (x_j^*(s^{(k)}))^2 \right) \right]^+ \tag{19}$$

where $[.]^+$ is defined as $[.]^+ = \max(., 0)$. The value of s is increased or decreased with the sign of $P - \sum_{i=0}^{|\mathcal{U}|-1} \sum_{j=0}^{M-1} p_{ij}^*(s^{(k)}) \cdot (x_j^*(s^{(k)}))^2$. The process stops when the power constraint is fulfilled. The proposed algorithm is summarized in table 2. Obviously, when constellation symbols are constrained to the values of a standard constellation, (P2) which is defined in table 2 will not be used. Similarly, when P_{UX} is uniform, (P1) is not used. An alternative interpretation of this algorithm is to recognize that $L(P_{UX}, x_0, .., x_{M-1}, s)$ is the Lagrangian dual of problem 9. Eq. (17-18) is an iterative method for solving

$$f(s) = \max_{P_{UX}, x_0, .., x_{M-1}} L(P_{UX}, x_0, .., x_{M-1}, s) \tag{20}$$

The dual optimization problem $\min_{s.t.\ s \geq 0} f(s)$ is solved in (19) with a gradient-type algorithm. Since $f(s)$ is convex [31], a gradient-search method is guaranteed to converge to a global optimum.

5 Result analysis

5.1 Point to point channel

We present in this section, the results of maximizing achievable rates for PtP case using M-PAM constellations with M=4, 8, 16 and for different values of SNR. To evaluate the contribution of constellation shaping, we compare, for a fixed SNR, the maximal achievable rate calculated by the algorithm proposed in the previous section to the "standard constellation" rate, whose symbols are used with equal probability, at the same *SNR* in terms of *SNR* saving (called *SNR* shaping gain). The *SNR* shaping gain depicted in Fig. (4) is the gain obtained with a fully optimized constellation ($P_{\mathcal{X}}$ and $\mathcal{X}$) compared to the standard M-PAM constellation and when symbols are used with the same probability. To avoid the complexity of constructing nearly optimal input distribution codes, another method for doing constellation shaping is to optimize only the position of symbols in the constellation. Each signal point is assumed to be chosen with the same probability however the position of each point in the constellation is optimized. The corresponding shaping gain is given in Fig. (5). We observe the following. The shaping gain depends on the *SNR* and on the size of the constellation. The maximum gain is obtained for mid-range *SNR*. The distribution of probability $P_{\mathcal{X}}$ (not reported) is very similar to the sampling of a gaussian distribution. With the half-optimized constellation ($\mathcal{X}$ only), a significant degradation is observed for mid range *SNR* compared to the fully optimized constellation. Hence, we can conclude that symbol pdf optimization is useless at low and high *SNR* whereas the fully-optimized constellation is efficient for mid-range *SNR*, in which case the gain increases with the size of the constellation.

5.2 Broadcast channel

Current broadcast systems are using two practical transmission schemes for sending information to users: orthogonal schemes in which the time and/or frequency is split between the users and superposition modulation schemes where the constellation for each user is fixed. In this section, a comparison is provided between these standard schemes and various (more complex) transmission strategies such as superposition coding. The effect of constellation shaping is evaluated by analyzing the achievable rate region curves obtained for an M-PAM constellation (M=4, 8, 16) and for several pairs (SNR_1, SNR_2). The following schemes are considered:

- Time Sharing using standard M-PAM (TS).
- Superposition Modulation (SM) - 3 possible configurations (see table 1)
- Superposition coding (SC) - 4 possible configurations (see table 1)

In the following, we denote by the "case 1" of superposition modulation when $M_1 = 2, M_2 = 4$ and when $M_1 = 2, M_2 = 8$. The "case 2" is when $M_1 = 4, M_2 = 2$ and when $M_1 = 4, M_2 = 4$. The "case 3" refers to the case when $M_1 = 8, M_2 = 2$.

Achievable rate region curves are provided in Fig. 6-11 for $M = 4, 8, 16$. For each value of M, the display of the results is limited to two different pairs of *SNR*. In complement with the achievable rate region curves, comparisons are also conducted in terms of *SNR* savings for target achievable rates (Maximum Shaping Gain) and in terms of Maximum Percentage of Gain for user 1. These two quantities are defined below.

Definition 1 *Consider two transmission strategies (A and B). The pair of rates* $(R_1 + R_2, R_2)$ *is achieved for* (SNR_1, SNR_2) *with A and for* $(SNR_1 + \Delta SNR, SNR_2 + \Delta SNR)$ *with B. The shaping gain (with A compared to B) is* ΔSNR*. The maximum shaping gain is defined as:*

$$MG_{SNR_{dB}}(A|B) = max_{R_2} \Delta SNR \tag{21}$$

Definition 2 *Consider two transmission strategies (A and B). For a given pair of SNR* (SNR_1, SNR_2) *and a fixed value of* R_2*, the achievable pair of rates is* $(R_1^A + R_2, R_2)$ *resp.* $(R_1^B + R_2, R_2)$ *with A resp. B. The gain on the achievable rate for user* 1 *is given by*

$$G_{R_1}(A|B) = \frac{(R_1^A + R_2) - (R_1^B + R_2)}{R_1^B + R_2} \cdot 100\ (\%) \tag{22}$$

The maximum gain on the achievable rate for user 1 *(with A compared to B) is given by*

$$MG_{R_1}(A|B) = max_{R_2} G_{R_1}(A, B) \tag{23}$$

5.2.1 Superposition modulation

In this section, the three possible configurations of Superposition Modulation are compared. We can see from Fig. 6 to 11 that $SM_{\mathcal{X},\overline{P_{UX}},\overline{P_X}}$ (optimization of $\mathcal{X}$ only) outperforms $SM_{\overline{\mathcal{X}},P_{UX},P_X}$ (optimization of P_{UX} only) in terms of maximal achievable rates per user when $M = 4$. For $M = 8$ and 16, $SM_{\overline{\mathcal{X}},P_{UX},P_X}$ can achieve slightly higher rates than $SM_{\mathcal{X},\overline{P_{UX}},\overline{P_X}}$. The implementation of a system with constellation symbols with non-standard positions and generated with the same probability is less complex than the implementation of a system which generates symbols with non-uniform joint distribution of probability. Thus, $SM_{\overline{\mathcal{X}},P_{UX},P_X}$ does not seem to be of interest since it is not very efficient in terms of achievable rates and is more complex to implement.

Figures of achievable rate region show that an improvement can be obtained with $SM_{\mathcal{X},P_{UX},P_X}$ (full optimization) compared to $SM_{\mathcal{X},\overline{P_{UX}},\overline{P_X}}$ (optimization of $\mathcal{X}$ only) and depending on $\delta_{SNR} = SNR_1 - SNR_2$. Numerical values of the maximum gain in achievable rate (MG_{R_1}) and of the maximum *SNR* savings ($MG_{SNR_{dB}}$) are given in table 3. We observe the following. A slight gain in terms of achievable rates can be translated into a noticeable gain in terms of *SNR* saving. The maximum shaping gain increases with the constellation size. Thus, constellation shaping for SM strategy seems more useful for high values of M. The analysis of the optimal matrix P_{UX} (results not reported) leads to the conclusion that X_1 and X_2 are not independent in general when using finite-size constellations. We observe also that the maximum shaping gain for $SM_{\mathcal{X},P_{UX},P_X}$ versus $SM_{\mathcal{X},\overline{P_{UX}},\overline{P_X}}$ increases when δ_{SNR} decreases, independently of M. In particular full optimization (vs optimization of the symbol position) does not provide significant improvement for large *SNR* gap in SM strategy.

M	SNR_1	SNR_2	$MG_{SNR_{dB}}(A\|B)$	$MG_{R_1}(A\|B)$
4	10	8	0.39	7.46%
		6	0.17	3.51%
		4	0.05	1.77%
		2	0.01	0.38%
8	16	14	$0.71^{(M_1=4,M_2=2)}$	$20.17\%^{(M_1=4,M_2=2)}$
		12	$0.57^{(M_1=4,M_2=2)}$	$13.21\%^{(M_1=4,M_2=2)}$
		10	$0.41^{(M_1=4,M_2=2)}$	$13.07\%^{(M_1=2,M_2=4)}$
		8	$0.33^{(M_1=2,M_2=4)}$	$18.93\%^{(M_1=2,M_2=4)}$
16	18	16	$1.05^{(M_1=8,M_2=2)}$	$10.67\%^{(M_1=8,M_2=2)}$
		14	$0.87^{(M_1=8,M_2=2)}$	$11.54\%^{(M_1=8,M_2=2)}$
		12	$0.64^{(M_1=8,M_2=2)}$	$12.08\%^{(M_1=4,M_2=4)}$
		10	$0.49^{(M_1=8,M_2=2)}$	$19.53\%^{(M_1=4,M_2=4)}$

Table 3: Comparison of $SM_{\mathcal{X},P_{UX},P_X}$ (A) and $SM_{\mathcal{X},\overline{P_{UX}},\overline{P_X}})$ (B) with respect to $MG_{SNR_{dB}}$ and MG_{R_1}

5.2.2 Time-Sharing (TS) or Superposition Modulation (SM)?

This section compares two strategies (TS and SM) classically considered in broadcast systems. In Fig. 6 and 7 ($M = 4$), we observe that the achievable rate region can be split into 2 parts. Indeed, for small and large values of R_2, TS is better than SM. On the contrary, SM is better than TS for middle-range values of R_2. Under a given rate requirement for one user, we can thus determine the best transmission strategy. We can also observe that the region in which SM is better than TS becomes small for larger values of SNR_2. With $M = 8$ (Fig. 8 and 9), the area in which SM is better than TS increases (compared to $M = 4$) by considering the union of the two possible configurations for SM: $M_1 = 2$, $M_2 = 4$ (case 1) and $M_1 = 4$, $M_2 = 2$ (case 2). This is particularly true when δ_{SNR} increases. We also observe that TS can achieve higher rates than SM (case 1) for good SNR_2 values. Indeed, the maximum rate of user 2 with SM is the maximum individual rate for a 4-PAM constellation whereas it is the individual user rate achieved using standard 8-PAM in the TS case. For low SNR_2 values, optimized 4-PAM may achieve higher rate than standard 8-PAM thus SM becomes better in this interval. For a 16-PAM constellation (Fig. 10 and 11), SM is always better than TS for the studied pairs of (SNR_1, SNR_2). Table 4 shows the maximum percentage of improvement in achievable rate of user 1 by TS when using $SM_{\mathcal{X},P_{UX},P_X}$ (full optimization) strategy in the interval where $SM_{\mathcal{X},P_{UX},P_X}$ is better than TS. Clearly, the maximum percentage of improvement increases when δ_{SNR} increases and an important gain is obtained for high values of δ_{SNR} as in the case of $SNR_1 = \delta_{SNR} = 10dB$ for a 4-PAM where the percentage of gain on achievable rate of user 1 varies between 0 and 40.7%. For a 8-PAM constellation, the percentage of gain on achievable rate of user 1 varies between 0 and 30.21% when $SNR_1 = 16\ dB$ and $\delta_{SNR} = 8$

dB. For a 16-PAM, percentages of improvements can be up to 35.08% when $SNR_1 = 18dB$ and $\delta_{SNR} = 8dB$. We can conclude that SM is a better option than TS especially for large δ_{SNR} values. TS is optimal in the region where we want to maximize the rate of user 2 for good values of SNR_2 because the single user rate achieved by TS is the rate achieved using standard M-PAM constellation (the constellation is split between users with SM). Thus, SM seems more gainful than TS when we want to serve users with very diverse SNRs.

5.2.3 *Is Superposition Coding necessary?*

For the three constellations under consideration ($M = 4, 8, 16$), the maximal achievable rate region obtained by the optimal general case of superposition coding when we consider the general form of P_{UX} (SC) can achieve, depending on M and user SNRs, a large region of rate pairs ($R_1 + R_2,\ R_2$) that cannot be achieved neither by TS nor by SM. Even when we fully optimize SM ($SM_{\mathcal{X},P_{UX},P_X}$) we are far from maximal achievable rate region. Sometimes the maximal achievable rate region curve is very close or even coincides with the $SM_{\mathcal{X},P_{UX},P_X}$ achievable rate region in a pair of rates ($R_1^* + R_2^*,\ R_2^*$). This is the case when $SM_{\mathcal{X},P_{UX},P_X}$ is the optimal superposition coding in terms of achievable rates. We can see for example in Fig. 6 that the pair of rates ($R_1^* + R_2^* = 1.096$, $R_2^* = 0.531$ which corresponds to the optimal rate pair when we optimize the general case of SC for $\theta = 0.23$) is an intersection point with $SM_{\mathcal{X},P_{UX},P_X}$ achievable rate region.

We are interested now in the numerical evaluation of the gain in rate of user 1 ($R_1 + R_2$) when we use $SC_{\mathcal{X},P_{UX},P_X}$ (full optimization) compared to the best strategy between TS and SM. This gain ($MG_{R_1}(SC_{\mathcal{X},P_{UX},P_X}|TS \bigcup SM_{\mathcal{X},P_{UX},P_X})$ calculated in % is the distance between the limit of the maximal achievable rate region and the limit of the union of achievable rate regions of TS and $SM_{\mathcal{X},P_{UX},P_X}$. The results are reported in table 4. We observe that the part of the maximal achievable rate region which is unachievable by TS and SM, is bigger when M is small because we observe that for the case of 4-PAM we have one configuration for SM. However, we have two configurations of SM for 8-PAM constellation and three configurations for 16-PAM constellation. Thus when M increases, the union of achievable rates for all SM cases tends to the sets of achievable rates by the general superposition coding. Asymptotically, we know that when $M \to \infty$, $SM_{\mathcal{X},P_{UX},P_X}$ is the optimal superposition coding scheme because it allows to achieve the capacity region for two-user AWGN BC using Gaussian alphabet for each user. Thus the maximum gain in user 1 rate decreases when constellation order M increases. We observe also that the gain in achievable rates is high for high values of δ_{SNR}. On the other hand, the experiments show that by using the general superposition coding strategy with the constraint that symbols should be equiprobable ($SC_{\mathcal{X},P_{UX},\overline{P_X}}$), the loss is limited compared to the full optimization ($SC_{\mathcal{X},P_{UX},P_X}$), 4.84%, 7.66% and 3.94% for the simulated pairs of ($SNR_1,\ SNR_2$) when $M = 4$, 8 and 16 respectively. This means that we can use equiprobable symbols with, in general, a small loss in achievable rates. However, $SC_{\mathcal{X},P_{UX},\overline{P_X}}$ is not an interesting

M	SNR_1	SNR_2	$MG_{R_1}(A\|B)$	$MG_{R_1}(A\|C)$
4	10	8	6.13%	6.72%
		6	11.14%	11.65%
		4	18.50%	16.69%
		2	28.43%	18.9%
		0	40.70%	23.54%
8	16	14	$7.80\%^{(M_1=2,M_2=4)}$	7.89%
		12	$13.60\%^{(M_1=2,M_2=4)}$	11.43%
		10	$21.15\%^{(M_1=2,M_2=4)}$	14.96%
		8	$30.21\%^{(M_1=2,M_2=4)}$	14.71%
16	18	16	$10.36\%^{(M_1=2,M_2=8)}$	2.96%
		14	$16.42\%^{(M_1=4,M_2=4)}$	2.94%
		12	$24.68\%^{(M_1=4,M_2=4)}$	5.29%
		10	$35.08\%^{(M_1=4,M_2=4)}$	4.80%

Table 4: Comparison of $SM_{\mathcal{X},P_{UX},P_X}$ (A)vs TS (B). Comparison of $SC_{\mathcal{X},P_{UX},P_X}$ (A) vs $TS \bigcup SM_{\mathcal{X},P_{UX},P_X}$ (C).

case when $SM_{\mathcal{X},P_{UX},P_X}$ can achieve better rates since SM is less complex to implement than SC.

Moreover, with standard M-PAM symbols the two possible configurations ($SC_{\overline{\mathcal{X}},P_{UX},P_X}$ (optimization of P_{UX} and P_X) and $SC_{\overline{\mathcal{X}},P_{UX},\overline{P_X}}$ (optimization of P_{UX} only)) gives very similar results in most considered pairs of SNR. We also observe that the loss in maximum achievable rate experienced by user 1 with $SC_{\overline{\mathcal{X}},P_{UX},P_X}$ is less than 10% under the rate experienced with $SC_{\mathcal{X},P_{UX},P_X}$. Thus we can use standard values of symbol positions without loosing much on achievable rates.

In general one can conclude that fixing constellations of users (i.e. assigning labels to the constellation so that we distinguish between the bits intended for each user) is not optimal for coding and may result in important loss in terms of rates for systems using finite-size constellations especially for low-order constellations. A better solution is to determine the optimal alphabet of the auxiliary alphabet U which is not necessarily a constellation and then to generate the codewords x^n which are not necessarily the sum of two codewords (see paragraph 3.4).

6 Application : coverage extension

We first consider a transmission over a broadcast channel with finite size input alphabet. For simplicity of the illustration and without loss of generality, let us assume that the existing user alphabet belongs initially to a standard constellation whose symbols are used with equal probability. We assume that existing user is at distance d_0 from the sender achieving a rate R_0. Some information is also to be transmitted to an upgraded layer of users. The sender can use up to 16 symbols, then several transmission schemes can be used. We are interested in comparing the transmission schemes to serve the new user under two scenarios: either the new user is closer to the transmitter than the existing user or the new user is farther than the existing one. For a target rate R_0 that is fixed for the existing user and achievable using a standard M-PAM and equiprobable symbols, we are interested in determining the variation of the coverage's diameter ratio between the two layer of users as a function of the achievable rate by the upgraded user for various broadcast transmission strategies. We assume that $SNR \propto \frac{1}{d^2}$.

6.1 The sender can use up to 16 symbols

6.1.1 Scenario 1

In this scenario, the system consists initially of one layer of users. Now assume that the data information is also to be transmitted to a second layer of users with higher SNR. In the following, we keep the notation from the preceding section where the user with greater SNR is denoted by user 1. Thus, in this scenario the legacy receivers are denoted by user 2 which is at a distance d_2 from the transmitter and achieving a rate R_0 when the data is modulated using standard 4-PAM constellation and equiprobable symbols. The upgraded receivers are denoted by user 1 ($SNR_1 > SNR_2$). We intend that the good user receives more throughput than user 2 via the use of 16-PAM.

In this example SNR_2 is fixed to 10 dB. Initially, user 2's alphabet belongs to a 4-PAM standard constellation (see section 3.1) and the rate transmitted to user 2 is $R_0 = 1.582$ bits/ch. use.

Now, a new layer of users called user 1 is introduced in the system with $SNR_1 > SNR_2$. Our target is to provide the maximum bit rate to the new user without changing R_0 or d_0 and using a 16-PAM. By enlarging the constellation and optimizing the symbol positions and probability distribution, we ensure that the rate of the initial user will not decrease after introducing a new user.

Consider now the results for the following strategies which can achieve a positive private-message rate for user 1: time sharing using standard 16-PAM, $SM_{\mathcal{X},\overline{P_{UX}},\overline{P_X}} M_2 = 8/M_1 = 2$ (optimization of $\mathcal{X}$ only), $SM_{\mathcal{X},P_{UX},P_X} M_2 = 8/M_1 = 2$ (full optimization) and $SC_{\mathcal{X},P_{UX},P_X}$ (full optimization). Fig.12 illustrates the variation of d_1/d_2, which is the ratio of the diameter of the coverage area for user 1 over the diameter of the initial coverage area for user 2, as a function of the achievable rate for user 1 for a target rate $R_0 = 1.582$ for user 2.

Let assume for example that the new user is midway between the transmitter and user 2 ($d_1/d_2 = 0.5$). Fig.12 shows that the most simple case of superposition modulation ($SM_{\mathcal{X},\overline{P_{UX}},\overline{P_X}} M_2 = 8/M_1 = 2$) provides 16.3% more bit rate than time sharing for the new user. If we move immediately to a more complex case and optimize P_{UX} ($SM_{\mathcal{X},P_{UX},P_X} M_2 = 8/M_1 = 2$), a gain of 21% is obtained on the bit rate of user 1 comparing to time sharing. This gain on achievable rate for the new user is equivalent to a gain of 1dB on SNR_1 comparing to superposition modulation with uniform P_{UX}. However if we move to the most general case of superposition coding, it doesn't provide significant gain comparing to superposition modulation.

Now we assume that the new user is close to the transmitter such that $d_1/d_2 = 0.2$. We observe that the gain on the bit rate of user 1 using the simple case of superposition modulation increases to 45.7% comparing to time sharing. By moving to a more complex case ($SM_{\mathcal{X},P_{UX},P_X} M_2 = 8/M_1 = 2$), a gain of 47.8% is obtained on the bit rate of user 1 comparing to time sharing. We observe also that it is relevant in this case to move to the most general case of superposition coding since it provides a gain of 61.8% on the bit rate of user 1 comparing to time sharing.

Consequently, using superposition modulation provides always noticeable gain comparing to time sharing. The general case of superposition coding $SC_{\mathcal{X},P_{UX},P_X}$ is useful when user 1 is close to the transmitter but not when it is close to user 2.

6.1.2 Scenario 2

Initially, consider a system of one layer of users, denoted by user 1, at a distance d_1 from the transmitter and achieving a rate R_0. Moreover, the alphabet of user 1 belongs to a standard 8-PAM constellation. In this example, SNR_1 is fixed to 18 dB. Thus user 1 can achieve a rate $R_0 = 2.73$ $bits/ch.\ use$ in the initial situation. In this scenario, we want to serve a second layer of users denoted by user 2 which is farther to the transmitter than the existing user i.e. $SNR_2 < SNR_1$.

Achievable rates for user 2 are obtained at different distance d_2 from the transmitter and using various transmission strategies for a target rate of user 1 equal to R_0 and a coverage diameter for user 1 fixed to d_1. Fig.13 illustrates the variation of d_2/d_1, which is the ratio of the diameter of the coverage area for user 2 over the diameter of the initial coverage area for user 1, as a function of the achievable rate for user 2 when a target rate for user 1 is fixed to $R_0 = 2.73$ $bits/ch.\ use$.

We observe in Fig.13 that superposition modulation can always achieve better rates for user 2 than time sharing using 16-PAM. Let assume first that we want to increase the diameter of the coverage area for the new user (user 2) such that $d_2/d_1 = 4$. Time sharing provides a bit rate less than 0.06 $bits/ch.use$. The most simple case of superposition modulation ($SM_{\mathcal{X},P_{UX},P_X} M_2 = 2/M_1 = 8$) provides a significant improvement on the achievable rate for user 2 which is equal to 0.4 $bits/ch.use$ in this case. If we increase the complexity by optimizing the joint probability distribution P_{UX}, we obtain 35% more bit rate for user 2 comparing to superposition modulation with uniform P_{UX}. If we move to the general case of superposition coding, we gain only 10% on the bit rate of the new user comparing to superposition modulation (see table 5). However when the new layer of users is at distance $d_2 = 2.25\ d_1$, the general case of superposition coding provides a significant gain of 41% on the achievable rate of user 2 comparing to superposition modulation.

d_2/d_1	SNR_2	MG_{R_2}	M_2/M_1
1.2589	16	4.9416	8/2
1.4125	15	20.1521	4/4
1.5849	14	12.7522	4/4
1.7783	13	8.2192	4/4
1.9953	12	7.4536	4/4
2.2387	11	41.4993	2/8
2.5119	10	30.8293	2/8
2.8184	9	22.9121	2/8

d_2/d_1	SNR_2	MG_{R_2}	M_2/M_1
3.1623	8	16.7443	2/8
3.5481	7	12.6033	2/8
3.9811	6	10.5427	2/8
4.4668	5	10.3343	2/8
5.0119	4	11.7414	2/8
5.6234	3	16.0961	2/8
6.3096	2	22.8535	2/8
7.0795	1	32.6194	2/8

Table 5: Comparison of $SC_{\mathcal{X},P_{UX},P_X}$ and $SM_{\mathcal{X},P_{UX},P_X}$ M_2-PAM/M_1-PAM w.r.t the gain in achievable rate of user 2: MG_{R_2} (%), where SNR_1=18 dB

Consequently, the general case of superposition coding can bring significant gains comparing to superposition modulation depending on the diameter of coverage area for the new layer of users. For superposition modulation, optimizing the joint distribution of probability P_{UX} provides often significant shaping gains.

6.2 The cardinality of the existing user alphabet is kept fixed :

In this section, we study the scenarios 1 (and 2) supposing that the legacy receivers will continue working as in the initial situation, still using 4-PAM (8-PAM). The system consists initially to one layer of users at distance d_0 from the transmitter and achieving a rate R_0. Now we want to change the transmitter such that upgraded receivers closer (farther) in range will be able to decode a refinement (coarse) layer and using a 16-PAM constellation. Thus only time sharing with $M_1 = M_2 = 4$ ($M_1 = 8$, $M_2 = 2$) and superposition modulation strategies can be used. We aim to study how small the reduction in legacy coverage can be made depending on the rate of the refinement (coarse) information achieved by the upgraded users. Thus suppose that the legacy coverage can be reduced from d_0 to d_2 (from d_0 to d_1). We have studied this problem for $SNR_0 = 12$ dB and for $SNR_1 - SNR_2 = 4$ dB in scenario 1 (and for $SNR_0 = 16$ dB, $SNR_2 = 14$ dB in scenario 2). Figures 14 (and 15) represent the reduction in coverage d_2/d_0 (and d_1/d_0 respectively) as a function of the rate of the refinement R_1 (of the coarse R_2), while the rate achieved by the legacy receivers is kept fixed to its initial situation, *i.e.* R_0.

We observe in figures 14 (and 15) that the gain of superposition modulation strategies over time sharing becomes more important when d_2/d_0 (d_1/d_0) is small. These figures show that using superposition modulation when both symbol positions and P_{UX} are optimized, we gain around 5 % from the initial coverage comparing to the case of superposition modulation where symbols are used with equal probability. We can observe also that a reduction of only 10% and 20% in coverage area for the existing user can serve the upgraded user with a rate up to 20% and 35% (9% and 15%) from the rate achieved by the legacy users, using $SM_{\mathcal{X},\overline{P_{UX}},\overline{P_X}}$. Consequently, by using $SM_{\mathcal{X},\overline{P_{UX}},\overline{P_X}}$, the legacy receivers still use 4-PAM (8-PAM in scenario 2) and we can serve a new layer of users with an acceptable rate, a small reduction in coverage area and with less complexity comparing to $SM_{\mathcal{X},P_{UX},P_X}$.

7 Conclusion

In this work we considered the problem of maximizing the achievable rate region for power constrained AWGN broadcast channel of two users using M-PAM constellations. The achievable rate region are given for various transmission strategies. Maximal achievable rate region for superposition coding and superposition modulation are obtained using constellation shaping. An iterative algorithm was proposed to solve this optimization problem. Then the efficiency of several strategies are compared. For superposition modulation, results showed that constellation shaping seems more useful for high values of M. Moreover, the gain in using a complex case of superposition modulation increases when the SNR gap between users decreases. We observed also that superposition modulation outperforms time sharing in a large part of the achievable rate region. On the other hand, it is shown that using the general case of superposition coding can bring important gains comparing to classical schemes. We observed also that in the case of finite input alphabet, superposition modulation is not the optimal strategy as in the case of Gaussian input alphabets. Finally, in order to make clear that this paper provides useful tools for the system designer, we considered two scenarios of coverage areas and user alphabets where the systems served initially one layer of users. Then we propose to serve a second layer of users and we evaluate the achievable rate of the new layer depending on the broadcast strategy. To improve the system performance compared to time sharing, we can optimize the joint probability distribution and symbol positions of the superimposed modulations or consider the general case of superposition coding. In this work we showed that the optimization of probabilities was often useful but not always. However, superposition coding brings sometimes significant gains comparing to superposition modulation depending on the diameter of coverage area for the new layer of users.

This work can also be extended to two dimensional constellations like M-QAM and other channel models. The maximization achievable rates using various transmission strategies can be performed also using the proposed algorithm based on alternative maximization with respect to symbol positions and the joint distribution of probability.

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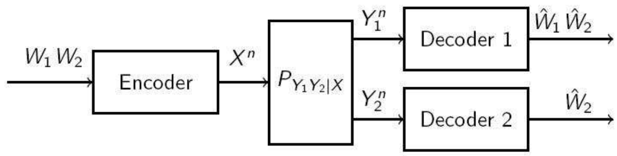

Figure 1: The two-user broadcast channel

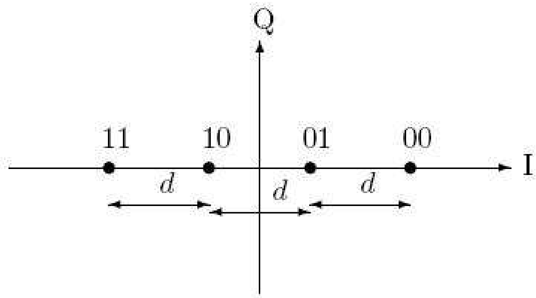

Figure 2: 4-PAM with equally spaced symbols

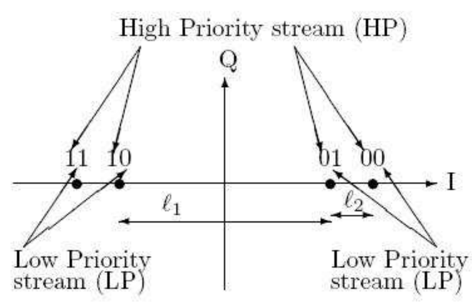

Figure 3: Hierarchical 4-PAM with parameter $\ell = \ell_1/\ell_2$

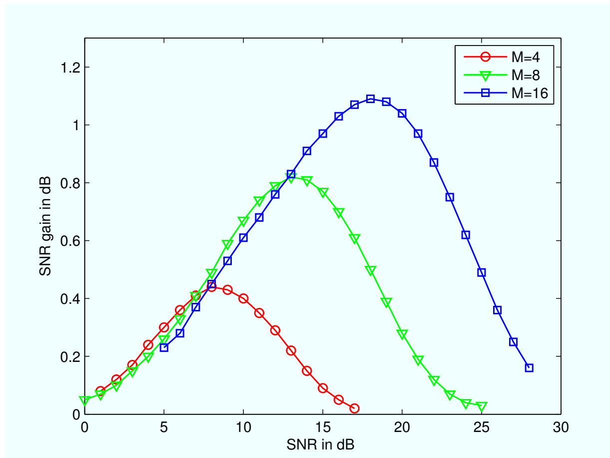

Figure 4: SNR shaping gain in dB vs PtP channel SNR - Optimal $\mathcal{X}$ and $P_{\mathcal{X}}$ vs Standard Constellation

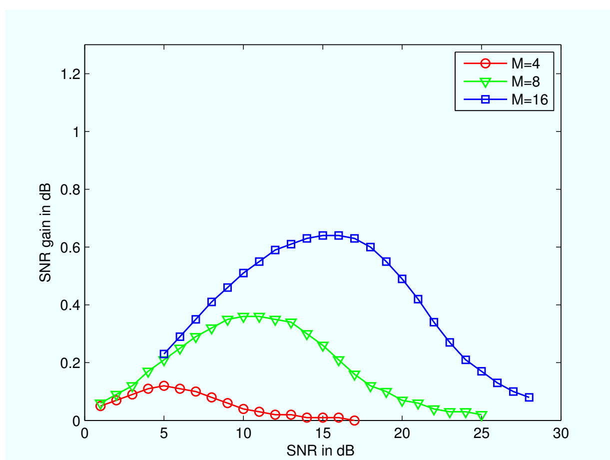

Figure 5: SNR shaping gain in dB vs PtP channel SNR - Optimal $\mathcal{X}$ vs Standard Constellation

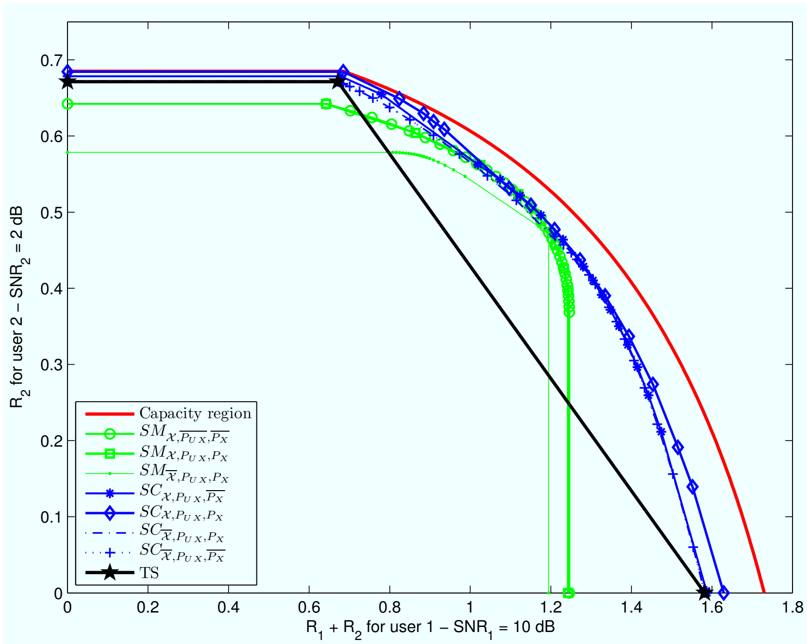

Figure 6: Achievable rate regions with $M = 4$ and $(SNR_1, SNR_2) = (10dB, 2dB)$

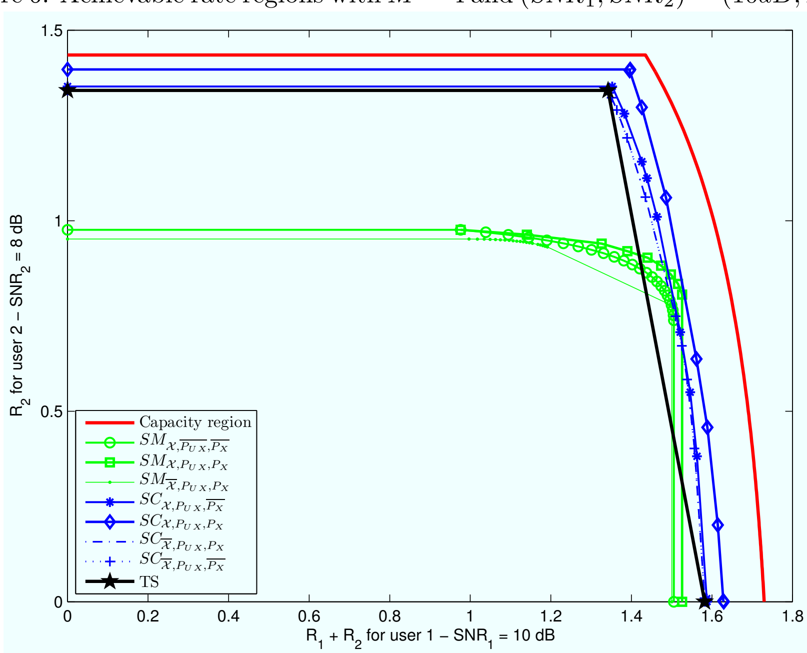

Figure 7: Achievable rate regions with $M = 4$ and $(SNR_1, SNR_2) = (10dB, 8dB)$

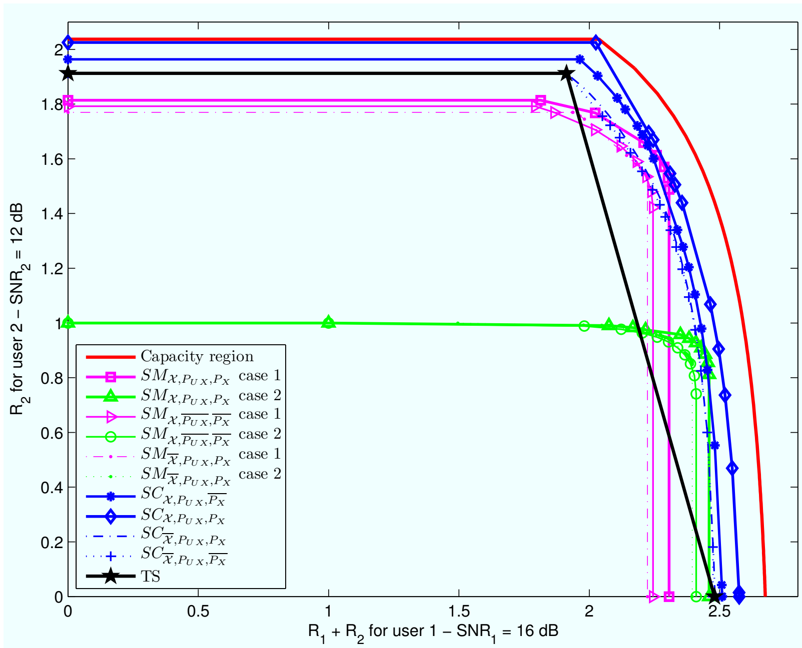

Figure 8: Achievable rate regions with $M = 8$ and $(SNR_1, SNR_2) = (16dB, 12dB)$

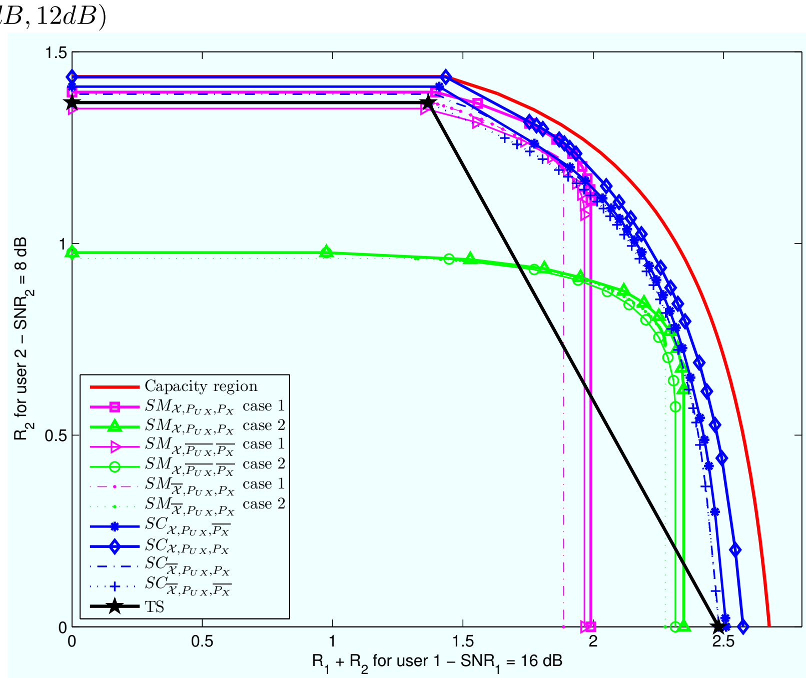

Figure 9: Achievable rate regions with $M = 8$ and $(SNR_1, SNR_2) = (16dB, 8dB)$

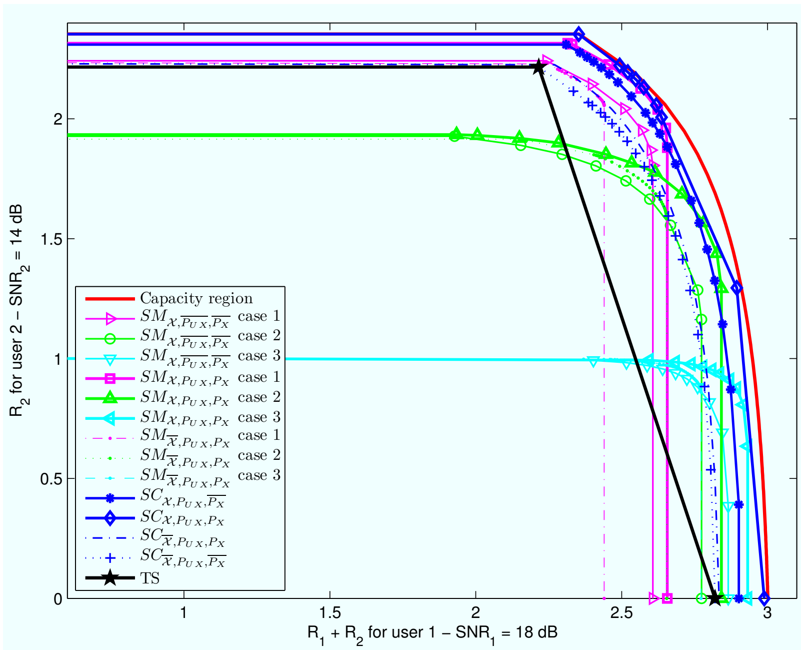

Figure 10: Achievable rate regions with $M = 16$ and $(SNR_1, SNR_2) = (18dB, 14dB)$

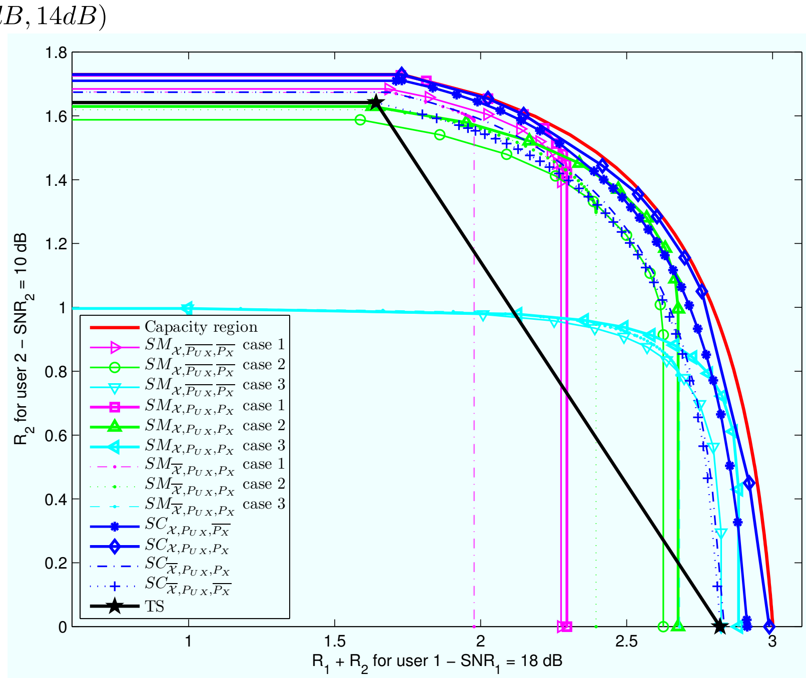

Figure 11: Achievable rate regions with $M = 16$ and $(SNR_1, SNR_2) = (18dB, 10dB)$

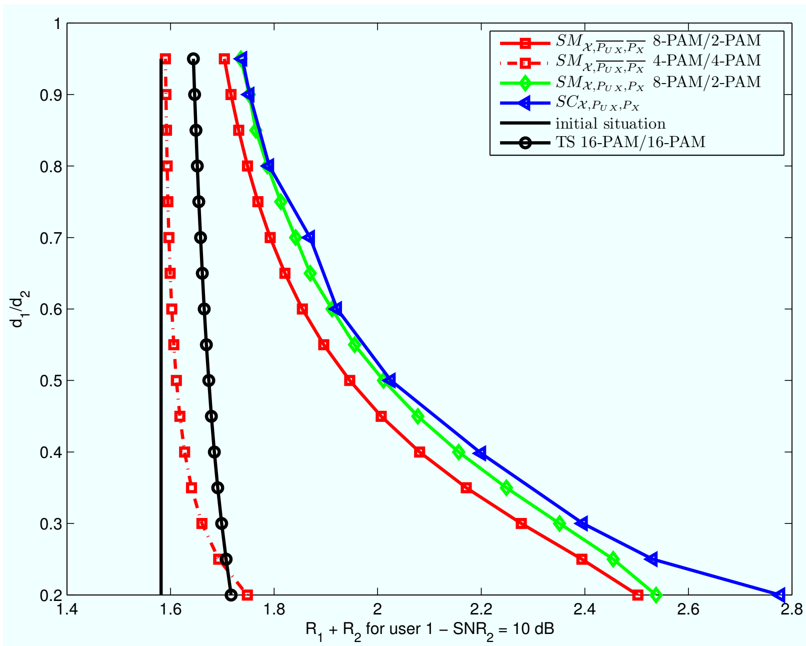

Figure 12: Coverage ratio d_1/d_2 as function of the achievable rate for user 1

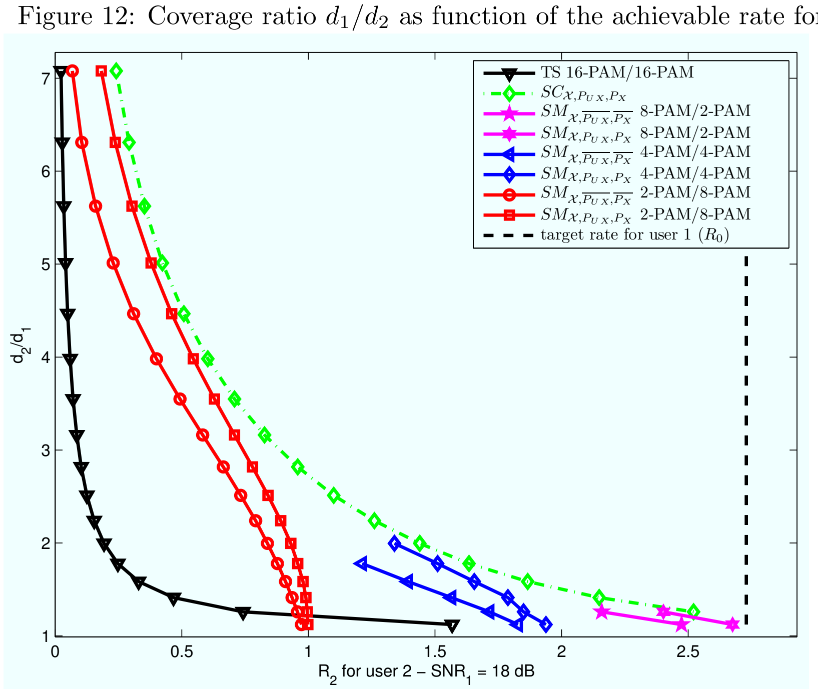

Figure 13: Coverage ratio d_2/d_1 as function of the achievable rate for user 2

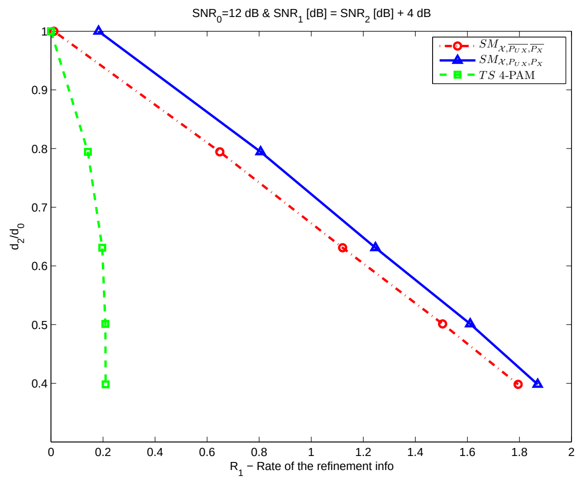

Figure 14: Reduction in legacy coverage d_2/d_0 in function of the rate of the refinement R_1.

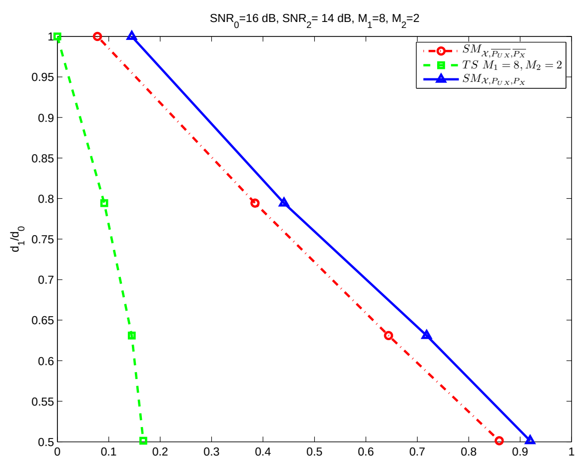

Figure 15: Reduction in legacy coverage d_1/d_0 in function of the rate of the coarse information R_2.